On the Skitovich-Darmois theorem for a-addic solenoids

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Abstract

Let X be a compact connected Abelian group. It is well-known that then there exist topological automorphisms α_j, β_j of X and independent random variables ξ_1 and ξ_2 with values in X and distributions μ_1, μ_2 such that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent, but μ_1 and μ_2 are not represented as convolutions of Gaussian and idempotent distributions. To put this in other words in this case even a weak analogue of the Skitovich-Darmois theorem does not hold. We prove that there exists a compact connected Abelian group such that if we consider three linear forms of three independent random variables taking values in X and the linear forms are independent, then at least one of the distributions is idempotent.

1 Introduction

The classical Skitovich-Darmois theorem states ([7],[1]): let $\xi_i, i = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables, and α_i, β_i be nonzero constants. Suppose that the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent. Then all random variables ξ_i are Gaussian.

This theorem was generalized to various classes of locally compact Abelian groups (see for example [2], where one can find references). In these researches random variables take values in a locally compact Abelian group X, and coefficients of the linear forms are topological automorphisms of X.

In particular, Feldman and Graczyk showed ([3]), that there not exists even a weak analogue of the Skitovich-Darmois theorem for compact connected Abelian groups. They proved the following: let X be an arbitrary compact connected Abelian group. Then there exist topological automorphisms $\alpha_i, \beta_i, i = 1, 2,$ of X and independent random variables $\xi_i, i = 1, 2,$ with values in X, such that the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ are independent. Whereas distributions of ξ_i are not convolutions of the Gaussian and idempotent distributions.

We show in this work that if we consider three linear forms of three random variables, then there exist compact connected Abelian groups, for which the weak analogue of the Skitovich-Darmois theorem holds. Namely, we will construct examples of compact connected Abelian groups, for which the independence of the three linear forms of independent random variables implies, that at least one random variable has idempotent distribution.

2 Definitions and designations

Let X be a second countable locally compact Abelian group. Let Λ be a set. If $X = X_\lambda$ for all $\lambda \in \Lambda$, then the direct product of the groups X_λ we denote by $X^{\mathfrak{n}}$, where $\mathfrak{n}$ is a cardinal number of the set Λ. Denote by $\aleph_0$ the cardinal number of a countable set. Denote by $Aut(X)$ the group of the topological automorphisms of X. Let k be an integer. Denote by f_k the mapping $f_k : X \to X$ defined by the equality $f_k x = kx$. Put $X^{(k)} = f_k(X)$.

Let $Y = X^*$ be the character group of X. The value of a character $y \in Y$ at $x \in X$ denote by (x, y). Let B a nonempty subset of X. Put

$$A(Y, B) = \{y \in Y : (x, y) = 1, x \in \ B\}.$$

The set $A(Y, B)$ is called the annihilator of B in Y. The annihilator $A(Y, B)$ ia a closed subgroup in Y. Let α be a topological endomorphism of X. For each $\alpha \in Aut(X)$ define the mapping $\tilde{\alpha} : Y \to Y$ by the equality $(\alpha x, y) = (x, \tilde{\alpha} y)$ for all $x \in X, y \in Y$. The mapping $\tilde{\alpha}$ is a topological endomorphism of Y. It is called an adjoint of α. The identity automorphism of a group denote by I.

In the paper we will use standard facts of Abstract harmonic analysis (see [6]). Let μ be a distribution on X. Put $x \in X$. Put $\bar{\mu}(M) = \mu(-M)$, where M is a Borel subset of X. The characteristic function of μ denote by equation

$$\hat{\mu}(y) = \int_X (x, y) d\mu(y), y \in Y.$$

Put $F_\mu = \{y \in Y : \hat{\mu}(y) = 1\}$. Then F_μ is a subgroup of Y, function $\hat{\mu}(y)$ is F_μ - invariant, i.e. $\hat{\mu}(y+h) = \hat{\mu}(y), y \in Y, h \in F_\mu$.

Denote by E_x the degenerate distribution, concentrated in x. Let K be a compact subgroup of X. Denote by m_K the Haar distribution on K. Denote by $I(X)$ the set of shifts of such distributions, i.e. the distributions of the form $m_K * E_x$, where K is a compact subgroup of X, $x \in X$. The distributions of the class $I(X)$ are called idempotent. The characteristic function of m_K has the form:

$$\hat{m}_K(y) = \begin{cases} 1, & y \in A(Y,K), \\ 0, & y \notin A(Y,K). \end{cases} \tag{1}$$

A distribution μ on the group X is called Gaussian if its characteristic function can be represented in the form

$$\hat{\mu}(y) = (x,y)\exp\{-\varphi(y)\}, \quad y \in Y,$$

where $\varphi(y)$ is a continuous nonnegative function satisfying equation

$$\varphi(u+v) + \varphi(u-v) = 2(\varphi(u) + \varphi(v)), \quad u, v \in Y.$$

Denote by $\Gamma(X)$ the set of Gaussian distributions on X.

3 Lemmas

In order to prove the main result we need some lemmas.

Lemma 3.1 *([5]). Let X be a second countable locally compact Abelian group, $\xi_i, i = 1, 2, \dots, n$, be independent random variables with values in X, and with distributions μ_i. Consider the linear forms $L_j = \sum_{i=1}^n \alpha_{ij}\xi_i$, where $\alpha_{ij} \in Aut(X)$. The linear forms $L_j, j = 1, 2, \dots, n$, are independent if and only if the following equation holds*

$$\prod_{i=1}^n \hat{\mu}_i(\sum_{j=1}^n \tilde{\alpha}_{ij}u_j) = \prod_{i=1}^n \prod_{j=1}^n \hat{\mu}_i(\tilde{\alpha}_{ij}u_j), \tag{2}$$

where $u_j \in Y, \tilde{\alpha}_{ij} \in Aut(Y)$.

The following lemma states that there exists the analogue of the Skitovich-Darmois theorem for finite Abelian groups, i.e. the independence of n linear forms implies, that all random variables have idempotent distributions.

Lemma 3.2 *([5]). Let X be a finite Abelian group. Let $\xi_i, i = 1, 2, \dots, n$, be independent random variables with values in X, and with distributions μ_i. Consider the linear forms $L_j = \sum_{i=1}^n \alpha_{ij}\xi_i$, where $\alpha_{ij} \in Aut(X), i, j = 1, 2, \dots, n$. The independence of the linear forms L_j implies that $\mu_i = E_{x_i} * m_K$, where K is a subgroup of X, $x_i \in X$, $i = 1, 2, \dots, n$.*

From lemmas 3.1 and 3.2 we obtain

Corollary 3.3 . *Let Y be a finite Abelian group. Let $\hat{\mu}_i(y), i = 1, 2, \dots, n$, $n \geq 2$, be characteristic functions on Y, satisfying equation (2), where $\tilde{\alpha}_{ij} \in Aut(Y), \tilde{\alpha}_{1j} = \tilde{\alpha}_{i1} = I, i, j = 1, 2, \dots, n$. Then $\hat{\mu}_i(y), i = 1, 2, \dots, n$, are the characteristic functions of the degenerate distributions.*

Below we will need some new notions. Denote by $\mathbb{Z}$ the infinite cyclic group, by $\mathbb{R}$ the additive group of real numbers, by $\mathbb{Z}(m)$ the group of residue modulo m, by $\mathbb{T}$ the circle group, by $\mathbb{Q}$ the group of rational numbers, by $\Delta_{\mathbf{a}}$ the group of **a**-addic numbers.

Let $\mathbf{a} = (a_0, a_1, \dots, a_n, \dots)$ be a fixed but arbitrary infinite sequence of natural numbers, where all $a_i > 1$. Consider the group $\mathbb{R} \times \Delta_{\mathbf{a}}$. Let B be a subgroup of $\mathbb{R} \times \Delta_{\mathbf{a}}$ of the form $B = \{(n, n\mathbf{u})\}_{n=-\infty}^{\infty}$, where $\mathbf{u} = (1, 0, \dots, 0, \dots)$. The factor-group $\mathbf{\Sigma_a} = (\mathbb{R} \times \Delta_{\mathbf{a}})/B$ is called an **a**-addic solenoid. The group $\mathbf{\Sigma_a}$ is a compact connected Abelian group with dimension 1. The character group of $\mathbf{\Sigma_a}$ is some subgroup of $\mathbb{Q}$.

The following lemma states that there exists the weak analogue of the Skitovich-Darmois theorem for the circle group. We assume that the characteristic functions of the random variables do not vanish.

Lemma 3.4 *[4]. Assume that $X = \mathbb{T}, \alpha_{ij} \in Aut(X), i, j = 1, 2, 3$. Let $\xi_i, i = 1, 2, 3$, be independent random variables with values in X and with distributions μ_i, such that their characteristic functions do not vanish. Suppose that $L_j = \sum_{i=1}^3 \alpha_{ij}\xi_i = 1, 2, 3$ are independent. Then μ_i are degenerate distributions.*

From lemmas 3.1 and 3.4 we obtain

Corollary 3.5 . *Assume that $Y = \mathbb{Z}$. Let $\hat{\mu}_i(y), i = 1, 2, 3$, $n \geq 2$, be nonnegative characteristic functions on Y satisfying equation*

$$\hat{\mu}_1(u_1 + u_2 + u_3)\hat{\mu}_2(u_1 - u_2 - u_3)\hat{\mu}_3(u_1 + u_2 - u_3) =$$

$$= \hat\mu_1(u_1)\hat\mu_1(u_2)\hat\mu_1(u_3)\hat\mu_2(u_1)\hat\mu_2(-u_2)\hat\mu_2(-u_3)\hat\mu_3(u_1)\hat\mu_3(u_2)\hat\mu_3(-u_3), u_i \in Y, i = 1,2,3. \quad (3)$$

Then $\hat\mu_i(y), i = 1,2,3,$ are characteristic functions of the degenerate distributions.

Lemma 3.6 *[2]. Let X be a compact Abelian group. Suppose that there exists an automorphism $\delta \in Aut(X)$ and an element $\tilde y \in Y$, such that the following conditions are satisfied:*

i)$Ker(I - \tilde\delta) = \{0\}$;

ii)$(I - \tilde\delta)Y \cap \{0; \pm\tilde y, \pm 2\tilde y\} = \{0\}$;

iii) $\tilde\delta\tilde y \neq -\tilde y$.

Then for any $n \geq 2$ there exist identically distributed random variables $\xi_i, i = 1,2,\dots,n,$ with values in X and with distribution $\mu \notin \Gamma(X) * I(X)$, such that the linear forms $L_j = \xi_1 + \sum_{i=2}^n \delta_{ij}\xi_i, j = 1,2,\dots,n,$ where $\delta_{ij} = I, i \neq j, \delta_{ii} = \delta$, are independent.

Note that compact connected Abelian groups satisfying the relation $f_p \in Aut(X)$ for any p are topologically isomorphic to a group of the form:

$$(\mathbf{\Sigma_a})^{\mathfrak{n}}, \mathbf{a} = (2,3,4,\dots), \mathfrak{n} \leq \aleph_0. \quad (4)$$

The following lemma for $n = 2$ was proved in the [3].

Lemma 3.7 . *Let X be a compact Abelian group, such that $f_2 \in Aut(X)$. Then there exist independent random variables $\xi_i, i = 1,2,\dots,n,$ with distributions $\mu_i \notin I(X) * \Gamma(X)$, and automorphisms $\alpha_{ij} \in Aut(X)$, such that the linear forms $L_j = \sum_{i=1}^n \alpha_{ij}\xi_i, j = 1,2,\dots,n,$ are independent.*

Proof. Two cases are possible: inclusion $f_p \in Aut(X)$ holds for any prime number p, and there is some prime number p such that $f_p \notin Aut(X)$

1. Consider the first case. By the note made above the group X is topologically isomorphic to the group of form (4). It is obvious that it suffices to prove the theorem for a group of the form $X = \mathbf{\Sigma_a}, \mathbf{a} = (2,3,4,\dots)$. Then group Y is topologically isomorphic to the group $\mathbb{Q}$. Let p and q be different prime numbers. Let H be a subgroup of Y of the form $H = \{\frac{m}{q^k}\}_{m,k\in\mathbb{Z}}$. Put $G = H^*, K = A(G, H^{(p)})$. Note that since numbers p and q are relatively prime, then $H \neq H^{(p)}$. Let λ be an arbitrary not-idempotent distribution on

G with support K. It is easy to see that the characteristic function $\hat{\lambda}$ is of the form

$$\hat{\lambda}(y) = \begin{cases} 1, y \in H^{(p)}, \\ c, y \notin H^{(p)}, \end{cases} \tag{5}$$

where $-1 < c < 1$.

Consider a function on Y

$$g(y) = \begin{cases} \hat{\lambda}(y), y \in H, \\ 0, y \notin H. \end{cases} \tag{6}$$

From the theorem 2.12 it follows that $g(y)$ is a positive-defined function. The Bohner theorem implies that there exist a distribution $\mu \in M^1(X)$ such that $\hat{\mu}(u) = g(y)$. It is obvious that $\mu \notin I(X) * D(X)$.

From (5) it follows that

$$g(y + pt) = g(y), \quad y, t \in H. \tag{7}$$

Let ξ_i be independent random variables with distribution μ.

Put $s = p^2 + q$. From the conditions of the theorem it follows that $s \in Aut(X)$. Let us show that the linear forms

$$L_1 = \xi_1 + p\xi_2 + p\xi_3 + ... + p\xi_n$$

$$L_2 = p\xi_1 + s\xi_2 + p^2\xi_3 ... + p^2\xi_n$$

$$L_3 = p\xi_1 + p^2\xi_2 + s\xi_3 ... + p^2\xi_n$$

....

$$L_n = p\xi_1 + p^2\xi_2 + p^2\xi_3 + ... + s\xi_n$$

are independent. By the lemma 3.1 it is enough to show that the following equation holds:

$$\hat{\mu}(u_1 + pu_2 + pu_3 + \cdots + pu_n)\hat{\mu}(pu_1 + su_2 + p^2u_3 + \cdots + p^2u_n) \cdots \hat{\mu}(pu_1 + p^2u_2 + \cdots + su_n) =$$

$$= \hat{\mu}(u_1)\hat{\mu}(pu_2)\hat{\mu}(pu_3) \cdots . \tag{8}$$

Using (7) it is easy to show that if $u_i \in H$, then equation (8) becomes a equality. So it is enough to consider the case when $u_i \notin H$ for some i. It is easy to see that in this case the right-hand side of equation (8) vanishes.

Let us show that the left-hand side of equation (8) vanishes too. Assume the converse, i.e. that the left-hand side of equation (8) does not vanish. Then the following system of equations holds:

$$\begin{cases} u_1 + pu_2 + pu_3 ... + pu_n = h_1, \\ pu_1 + su_2 + p^2u_3 ... + p^2u_n = h_2, \\ .... \\ pu_1 + p^2u_2 + p^2u_3 ... + su_n = h_n, \end{cases} \tag{9}$$

where $h_i \in H$.

Add the first equation of the system (9) multiplied by $(-p)$ to the each equation of the system (9) starting from the second. We obtain that $qu_i = h_i - h_1, i = 2, 3, \ldots, n$. Thus $u_i \in H, i = 2, 3, \ldots, n$. From this and from the first equation of the system (9) we have that $u_1 \in H$. Finally we obtain that $u_i \in H, i = 1, 2, \ldots, n$. This contradicts the assumption.

2. Assume that for some prime p the following inclusion holds

$$f_p \notin Aut(X) \tag{10}$$

Suppose that p is the smallest from the prime numbers satisfying condition (10). Since X is a connected group then $X^{(n)} = X$ for all natural n. Hence if $f_p \notin Aut(X)$, then $Ker f_p \neq \{0\}$.

From the condition of the theorem it follows that $p \geq 3$. Put $a = 1 - p$. Since p is a smallest natural number satisfying condition (10), then $f_{-a} \in Aut(X)$. Hence $f_a \in Aut(X)$. By the well-known theorem $Ker f_p = A(X, Y^{(p)})$. It implies that $Y^{(p)} \neq Y$. Put $\tilde{y} \in Y^{(p)}$ and verify that the automorphism $\delta = f_a$ and the element $\tilde{y}$ satisfy to conditions of lemma 3.6. We have $\tilde{f}_a = f_a$ and $I - \tilde{f}_a = \tilde{f}_p$. Since Y is torsion-free then $Ker(I - \tilde{f}_a) = \{0\}$, i.e. condition (i) holds. Thus $(I - \tilde{f}_a)Y = Y^{(p)}$. Since $p \geq 3$, then numbers 2 and p are relatively prime. Hence there are integers m and n such that $2m + pn = 1$. Thus $y = 2my + pny$. So if $\tilde{y} \notin Y^{(p)}$, then $2\tilde{y} \notin Y^{(p)}$ too. It implies that condition (ii) holds. Since the group Y is torsion-free, then it is obvious that condition (iii) holds. We use 3.6 and obtain the needed result.■

4 Main theorem

The proof of the main theorem is bulky. So it is divided into two parts. In the first part we use corollaries 3.3 and 3.5, in the second part we use lemma

3.7.

Theorem 4.1 . *Assume that* $X = \mathbf{\Sigma}_a$. *Then there are two cases:*

1) For any prime number p *the relation* $f_p \notin Aut(X)$ *holds. Let* $\xi_i, i = 1, 2, 3,$ *be independent random variables with values in* X *and with distributions* μ_i. *Then the independence of the linear forms* $L_j = \sum_{i=1}^{3} \alpha_{ij}\xi_i$, $\alpha_{ij} \in Aut(X), i, j = 1, 2, 3,$ *implies that at least one distribution* $\mu_i \in I(X)$.

2) There exists a prime number p *such that* $f_p \in Aut(X)$. *Then there are independent random variables* $\xi_i, i = 1, 2, 3,$ *with values in* X *and with distributions* $\mu_i \notin \Gamma(X) * I(X)$, *and automorphisms* $\alpha_{ij} \in Aut(X)$, *such that the linear forms* $L_j = \sum_{i=1}^{3} \alpha_{ij}\xi_i, j = 1, 2, 3,$ *are independent.*

Proof.

1. Suppose that $f_p \notin Aut(X)$ for any p. It means that $Aut(X) = \{I, -I\}$. It is easy to show that the general problem can be reduced to the case when

$$\begin{aligned} L_1 &= \xi_1 + \xi_2 + \xi_3, \\ L_2 &= \xi_1 - \xi_2 + \xi_3, \\ L_3 &= \xi_1 - \xi_2 - \xi_3. \end{aligned} \tag{11}$$

From lemma 3.1 the independence of the linear forms (11) implies, that equation (3), where Y is a subgroup of $\mathbb{Q}$, holds.

Put $N_i = \{y \in Y : \hat{\mu}_i(y) \neq 0\}, N = \cap_{i=1}^{3} N_i$. From (3) we infer, that N is a subgroup in Y. Moreover, from (3) it is easy to see, that N has a property: in $2y \in N$, then $y \in N$. There are two cases: $N \neq \{0\}$ and $N = \{0\}$.

Also note that the following relation holds: $Y = Y^{(2)} \cup (\tilde{y} + Y^{(2)})$, where $\tilde{y} \notin Y^{(2)}$.

A. Assume that $N \neq \{0\}$. Let us prove an useful equation. Suppose that t_1 and t_2 belong to a same coset of the factor-group $Y/Y^{(2)}$. Then the following equation holds:

$$|\hat{\mu}_{i_1}(t_1)||\hat{\mu}_{i_2}(t_2)||\hat{\mu}_{i_3}(t_2)| = |\hat{\mu}_{i_1}(t_2)||\hat{\mu}_{i_2}(t_1)||\hat{\mu}_{i_3}(t_1)| \tag{12}$$

where all i_j are pairwise different. Indeed, there exist $\hat{u}_1$ and $\hat{u}_2$, such that $\hat{u}_1 + \hat{u}_2 = t_1, \hat{u}_1 - \hat{u}_2 = t_2$. Putting $u_1 = \hat{u}_1, u_2 = \hat{u}_2, u_3 = 0$ in (3), $u_1 = \hat{u}_1, u_2 = -\hat{u}_2, u_3 = 0$ in (3), and equating the results, we obtain:

$$|\hat{\mu}_1(t_1)||\hat{\mu}_2(t_2)||\hat{\mu}_3(t_1)| = |\hat{\mu}_1(t_2)||\hat{\mu}_2(t_1)||\hat{\mu}_3(t_2)|,$$

from what it follows equation (12).

Put $\nu_i = \mu_i * \bar{\mu}_i, i = 1, 2, \dots, n$. Then $\hat{\nu}_i(y) = |\hat{\mu}_i(y)|^2, y \in Y$, functions $\hat{\nu}_i(y)$ are nonnegative and satisfy equation (2). It is suffice to show that $\hat{\nu}_i(y)$ are characteristic functions of the idempotent distributions. From this we will obtain that $\hat{\mu}_i(y)$ also are characteristic functions of the idempotent distributions.

First show that for any $y \in N$ the equality $\hat{\nu}_i(y) = 1, i = 1, 2, 3$, holds. Put $y_0 \in N$. Consider a subgroup H of Y generated by y_0. Note that $H \cong \mathbb{Z}$. Consider the restriction of equation (3) to the subgroup H. From corollary 3.5 we obtain, that $\nu_i = E_{x_i}, x_i \in X, i = 1, 2, 3$. It means that $\hat{\nu}_i(y) = 1, i = 1, 2, 3, y \in H$. Since $H = \langle y_0 \rangle$ and y_0 was arbitrarily chosen from N, then $\hat{\nu}_i(y) = 1, i = 1, 2, 3, y \in N$.

Now we will show that $N_i = N$. Assume that $N_i, i = 1, 2, 3$, do not coincide. Then there are $y_1 \in N_1, y_1 \notin N_j$, where j is equal to either 2 or 3. Put $t_1 = y_1, t_2 = y_2$, where $y_2 \in N$ and y_1,y_2 belong to the same coset of the factor-group $Y/Y^{(2)}$, in (12). We can make such choice. Indeed, on the one hand $N \cap Y^{(2)} \neq \{0\}$ because N is a subgroup. From the other hand $N \cap (\tilde{y} + Y^{(2)}) \neq \{0\}$. It follows from the property of the subgroup N: if $2y \in N$, then $y \in N$. We infer that the left-hand side of equation (3) is equal to nonnegative number, and the right-hand side of equation (3) is equal to zero. This is a contradiction. So we have that $N_i = N, i = 1, 2, 3$.

Taking in the attention that the characteristic functions $\hat{\nu}_i(y)$ are N-invariant, consider the equation induced by the equation (3) on the factor-group Y/N. Put $f_i([y]) = \hat{\nu}_i([y])$. Since Y/N is a finite Abelian group then from corollary 3.3 we infer that $f_i([y])$ are characteristic functions of some idempotent distributions. Returning to the original equation from the induced equation, we obtain needed result.

B. Consider the case $N = \{0\}$.

Put first $u_2 = 0, u_3 = u_1 = y$, after $u_3 = 0, u_1 = u_2 = y$, and finally $u_1 = 0, u_2 = u_3 = y$ in (3), we infer:

$$\hat{\mu}_1(2y) = \hat{\mu}_1^2(y)|\hat{\mu}_2(y)|^2|\hat{\mu}_3(y)|^2, \quad y \in Y. \tag{13}$$

$$\hat{\mu}_2(2y) = |\hat{\mu}_1(y)|^2\hat{\mu}_2^2(y)|\hat{\mu}_3(y)|^2, \quad y \in Y. \tag{14}$$

$$\hat{\mu}_3(2y) = |\hat{\mu}_1(y)|^2|\hat{\mu}_2(y)|^2\hat{\mu}_3^2(y), \quad y \in Y. \tag{15}$$

Show that

$$\hat{\mu}_i(2y) = 0, y \in Y, y \neq 0, i = 1, 2, 3. \tag{16}$$

Indeed, assume the converse, i.e. there are $y_0 \in Y, y_0 \neq 0$ and i_0, such that $\hat{\mu}_{i_0}(2y_0) \neq 0$. Then from equalities (13)-(15) it follows that $\hat{\mu}_i(y_0) \neq 0, i = 1, 2, 3$. Hence $N \neq \{0\}$, that contradict the assumption.

Show that at least one distribution μ_i is idempotent. Assume the controversial. From representation(1) it follows that there exist $y_1 \neq 0, y_2 \neq 0, y_3 \neq 0$, such that

$$\hat{\mu}_1(y_1)\hat{\mu}_2(y_2)\hat{\mu}_3(y_3) \neq 0. \tag{17}$$

From equality (16) it follows that $y_i \in \tilde{y} + Y^{(2)}$.

Solve the system of equations

$$\begin{cases} u_1 + u_2 + u_3 = y_1, \\ u_1 - u_2 - u_3 = y_2, \\ u_1 + u_2 - u_3 = y_3. \end{cases} \tag{18}$$

For a element $y_0 \in Y^{(2)}$ denote by $\frac{y_0}{2}$ such element of Y, that $2\frac{y_0}{2} = y$. Note, that for any elements $h_1, h_2 \in \tilde{y} + Y^{(2)}$ the following inclusion $h_1 + h_2 \in Y^{(2)}$ holds. From this and from $y_i \in \tilde{y} + Y^{(2)}, i = 1, 2, 3$, we infer that there exist a solutions of the system (18) and they has a form

$$\begin{cases} u_1 = \frac{y_1+y_2}{2}, \\ u_2 = \frac{y_3-y_2}{2}, \\ u_3 = \frac{y_1-y_3}{2} \end{cases} \tag{19}$$

Put the solutions of (19) in equation (3) and taking into the account (17) we infer that the right-hand side of (3) is not equal to 0, whereas in particular it follows that

$$\hat{\mu}_1(\frac{y_1+y_2}{2})\hat{\mu}_2(\frac{y_3-y_2}{2})\hat{\mu}_3(\frac{y_1-y_3}{2}) \neq 0. \tag{20}$$

Reasoning the same way, as from (17) we obtained (20), form (20) we will obtain that

$$\hat{\mu}_1(\frac{y_1+y_3}{2})\hat{\mu}_2(\frac{y_1+y_2-2y_3}{2})\hat{\mu}_3(\frac{y_2+y_3}{2}) \neq 0. \tag{21}$$

It easy to verify that one of the numbers $\frac{y_1+y_3}{2}, \frac{y_1-y_3}{2}, \frac{y_3-y_2}{2}, \frac{y_3+y_2}{2}$ belongs to $Y^{(2)}$ and is not equal to 0. Denote it by $2y_0$. Then las two relationships imply that $\mu_{i_0}(2y_0) \neq 0$ for some i_0, what contradicts to (16).

2. Now consider the case $f_p \in Aut(X)$ for some prime p. If $f_2 \in Aut(X)$, then the statement follows from the lemma 3.7. Assume that $f_2 \notin Aut(X)$. Consider the function $\rho(x)$ on X defined by the equation

$$\rho(x) = 1 + Re(x, y_0).$$

Let μ be a distribution on X with the density $\rho(x)$ with respect to m_X. It is obvious that $\mu \notin \Gamma(X) * I(X)$. The characteristic function of the distribution μ has the form:

$$\hat{\mu}(y) = \begin{cases} 1, & y = 0, \\ \frac{1}{2}, & y = \pm y_0, \\ 0, & y \notin \{0, y_0, -y_0\}. \end{cases} \tag{22}$$

Let $\xi_i, i = 1, 2, \dots, n$, be independent random variables with the distribution μ. Consider the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3, L_2 = \xi_1 + p\xi_2 + \xi_3, L_3 = \xi_1 + \xi_2 + p\xi_3$. Let us show that $L_j, j = 1, 2, \dots, n$, are independent. By lemma 3.1 the linear forms L_j are independent if and only if the following equation holds:

$$\hat{\mu}(u+v+t)\hat{\mu}(u+pv+t)\hat{\mu}(u+v+pt) = \hat{\mu}^3(u)\hat{\mu}^2(v)\hat{\mu}^2(t)\hat{\mu}(pv)\hat{\mu}(pt). \tag{23}$$

We show that equation (23) holds. It is obvious, that it suffices to consider the case, when at least two of three elements u, v, t are not equal to 0. It is easy to see that in this case the right-hand side of equation (23) is equal to 0. Let us show that the left-hand side of equation (23) vanishes too.

Suppose that there are some elements u, v, t such that the left-hand side of equation (23) does not vanish. Then there exist some $h_i \in \{0, y_0, -y_0\}, i = 1, 2, 3$, such that u, v, t satisfy the system of equations

$$\begin{cases} u + v + t = h_1, \\ u + pv + t = h_2, \\ u + v + pt = h_3. \end{cases} \tag{24}$$

It is easy to obtain from (24) that

$$(p-1)v, (p-1)t \in \{0, \pm y_0, \pm 2y_0\}. \tag{25}$$

Relationship (25) can not holds because of $(p-1) = 4k$, but $y_0 \notin Y^{(2)}$. From this it follows that the left-hand side of equation (23) is equal to 0.

The second case can be considered in the same way. But we have to consider the linear forms $L_1 = \xi_1+\xi_2+\xi_3, L_2 = \xi_1-p\xi_2+\xi_3, L_3 = \xi_1+\xi_2-p\xi_3$.

The theorem is completely proved.

$\blacksquare$

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